PROGRAMMED CONTROL OF SYSTEMS WITH RANDOM PARAMETERS

(PROGRAMMNOE REGULIROVANIE SISTEM SO SLUCHAINYMI PARAMETRAMI)

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Let us assume that a controlled system is subjected to random impulses and that it has a corrective arrangement guaranteeing the carrying through of the process according to the given program, conditioned by the uninterrupted receipt of complete information about the random impulses at each given instant. If the information about the random impulse is transmitted with a distortion arising from the appearance of random errors, errors of measurement, the appearance of lags, inertia, etc., then the actual process will be different from the desired process. Below are given estimations of the distortions which bring about the deviation of the process from the given one, within the permissible limits. There is considered the case in which the actual motion is periodic and the case in which the actual motion is discontinuous.

The results can be treated as the conditions of stability of the motions, that is, as the conditions of preservation and stability of periodic motions in the presence of constantly acting random impulses.

In order to give uniformity to the results, the actual process is treated as if it were random. This article is the immediate continuation of previous work [1] in which the determinantal case is considered.

1. Consider the system of differential equations

$$\frac{dx}{dt} = f(x, t, \eta(t)) + u(t, \xi(t)) \tag{1.1}$$

where x is an n-dimensional vector; $\eta(t)$, $\xi(t)$ are random scalar functions; $f(x, t, \eta(t))$, $u(t, \xi(t))$ are n-dimensional vector functions.

There arises the problem of the choice of the functions $\xi(t)$, $u(t,\xi(t))$ so that the *a priori* given (and perhaps random) function $x = g(t)$ would be a solution of the system of equations (1.1). Obviously the simplest solution to this problem would yield the relations

$$\xi(i) = \eta(t), \qquad u(t, \xi(t)) = g'(t) - f(g(t), t, \eta(t)) \tag{1.2}$$

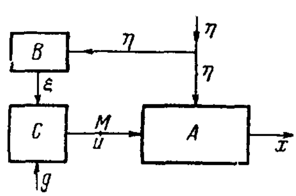
However, upon construction of the equation $u(t, \xi(t))$ it is possible to obtain incomplete information about the significance of the function $\eta(t)$ and moreover this information may be received with a certain lag. It may occur that the function $u(t, \xi(t))$ can be chosen only from a determined class of functions (for example, trigonometric polynomials), and consequently the second equation in (1.2) can only be fulfilled approximately. Consider

$$r(t, \eta(t), \xi(t)) = u(t, \xi(t)) - g'(t) - f(g(t), t, \eta(t)) \tag{1.3}$$

where $\xi(t)$ is a random function correlated with $\eta(t)$. Then values are to be found for the distortion $r(t, \eta(t), \xi(t))$, upon the fulfilment of which the deviation of the actual random process $x(t)$, described by system (1.1) from the given process $g(t)$, does not exceed the given magnitude.

The structural scheme of the system, described by the equation, is presented in Fig. 1: here A is the object, the actual required process; random impulses η act upon A; these impulses are sent simultaneously through C into a correcting arrangement B, the purpose of which is the working out of the corresponding controlling impulse.

Consider now certain general aspects related to the theory of differential equations with random parameters. It may be shown [2, p.30] that a random quantity can be determined as a measurable function, fixed in a certain region of values Ω (or a region of elementary events). Here two random functions $\eta(t)$ and $\xi(t)$ enter into Equation (1.1). If $\xi(t)$ is functionally connected with $\eta(t)$, then both of these functions have the same region of values Ω. If the function $\xi(t)$ is connected with $\eta(t)$ correlatively, that is, at a determined realization of $\eta(t)$, the function $\xi(t)$ will be a random function with a region of values Δ; then evidently as a general region of values it is possible to take the region being the product of regions Ω and Δ with a measure measure chosen in the corresponding manner.



In such a manner, it is possible to consider that all random magnitudes entering Equation (1.1) will be random magnitudes connected with the general region of values Ω.

If in a linear region of random magnitudes there is determined by some means a norm (as in a region of measurable functions determined in a region of values Ω), then the differential equation (1.1) is transformed into a differential equation given in the linear normalized space M the elements of which are random vectors. Further, as the initial vectors for the solution of Cauchy's problem, it is necessary to take not only the determinantal vectors but any other random vectors from M. Evidently, in the presented treatment of Equation (1.1), the integral is derived from a random function conforming to a scalar argument t; it should be understood how the integral is derived in the sense of Bokhner [3, p.59]. In particular, if in place of the square of the norm of the random vector one can take the mathematical probability of the square of the length of the vector, then the concept of the derivation and the integral of the random function coincides with that generally understood [4, p.214].

It should be noted that the theory of differential equations in linear normalized regions is well developed at the present time. Relying upon this theory, it is possible without difficulty to formulate the conditions of existence, of uniqueness, and of continuity of solutions [5], to consider the questions of stability [6], or the questions of the existence and determination of periodic solutions [7].

Making use of the above-mentioned reasoning, it is possible, obviously, to utilize the results of the cited works for the study of differential equations with random parameters.

2. Let $g(t)$ be a piecewise differentiable random process. Carrying out in system (1.1) a change of variable $z = x - g(t)$, there results

$$\frac{dz}{dt} = f(z + g(t), t, \eta(t)) + u(t, \xi(t)) - g'(t) \qquad (2.1)$$

Making use of (1.3) and introducing the designation

$$Z(z, t, \eta(t)) = f(z + g(t), t, \eta(t)) - f(g(t), t, \eta(t))$$

the system of equations of the perturbed motion takes on the form

$$\frac{dz}{dt} = Z(z, t, \eta(t)) + r(t, \eta(t), \xi(t)) \qquad (2.2)$$

The vector random function $r(t)$ determines the error resulting from the presence of distortions connected with the transfer of information about $\eta(t)$ into the correcting arrangement; the deviation of the random function from zero coincides with the deviation of the solution $x(t)$ of system (1.1) from the given function $g(t)$. As a measure of the deviation of the random magnitude z from zero, we take

$$\|z\| = (M\,|\,z\,|^2)^{1/2}, \qquad |\,z\,| = \max\,|\,z_i!\,|\;(1 \leqslant i \leqslant n) \tag{2.3}$$

Here M is the operation of finding the mathematical probability, z_i is the projection of vector z. If A is a matrix with elements a_{ik}, then it is assumed that

$$|\,A\,| = \max_{1 \leqslant i \leqslant n} \sum_{k=1}^{n} |\,a_{ik}\,|$$

Obviously, the inequality

$$|Az\,| \leqslant |\,A\,|\,|z| \tag{2.4}$$

is valid, whereby the equality can be attained. If matrix A and vector z are random, then the inequality (2.4) is justified for any separate realization, that is for the fixation of the element ω of the chosen region Ω.

We now separate from the function $Z(z,\ t,\ \eta(t))$, according to some rule, the linear part $A(t,\ \eta(t))z$, and Equation (2.2) is then written in the form

$$\frac{dz}{dt} = A\ (t,\ \eta\ (t))\ z + R\ (z,\ t,\ \eta\ (t)) + r\ (t,\ \eta\ (t),\ \xi\ (t)) \tag{2.5}$$

By D is designated the part of the region of random values from M, determined by the inequality $\|\,z\,\| < \epsilon$.

The following limits on the equation are assumed:

a) The values $M|A(t,\eta(t)\,\|^2$, $\|R\,(z,\ t,\ \eta(t))\,\|$, $\|\,r\,(t,\ \eta(t),\ \xi(t))\,\|$ are finite for almost all t.

b) The functions $|A(t,\ \eta(t))\,|$ and $|R(z,\ t,\ \eta(t))\,|$ for any fixed z may be integrated (as random functions) over any interval $[\,k\,T,\ (k+1)\ T]$, where k is a positive whole number and $T < 0$.

c) The function $R(z,\ t,\ \eta(t))$ satisfies the Lipschitz condition $\|R\,(z,\ t,\ \eta(t)) - R(y,\ t,\ \eta(t))\,\| \leqslant L\,\|\,z - y\,\|$, $(L = \text{const},\ z \subset D,\ y \subset D)$.

d) The function $\|\,r\,(t,\ \eta(t),\ \xi(t))\,\|$ is integrated (according to Lebegue) over any interval $[\,k\,T,\ (k+1)T]$.

e) There exists the fundamental matrix $W = W(t,\ \tau,\ \omega)$ of the system $x' = A(t,\ \eta(t))x$, such that

$$M_\tau\,|\,W\,(t,\ \tau,\ \omega)\,|^2 \leqslant B^2 e^{-2\alpha(t-\tau)} \qquad (B \geqslant 1,\ \alpha > 0)$$

Here M_τ is the operation of finding the conditional mathematical probability with $t \leqslant \tau$, $\eta(t) = \eta_0(t)$ where $\eta_0(t)$ is a certain realization of $\eta(t)$.

f)
$$\lambda = \alpha - LB > 0$$

g) The random magnitudes $|W(t, \tau, \omega)|$ and $|R(z(\tau), \dot{\tau}, \eta(\tau)$, just as the random magnitudes $|W(t, \tau, \omega)|$ and $|r(\tau, \eta(\tau), \xi(\tau))|$, with the physical value τ and the determined realization $\eta(t) = \eta_0(t)$ at $t \leqslant \tau$ are statistically independent.

The last condition means that the course of the process, described by system $x' = A(t, \eta(t))x$, does not depend upon the distortions, occurring at a given instant of time. In other words, in the case of a linear object A, this means that if the closed circuit shown in Fig. 1 is opened at point M, then the random magnitude u, on the output of C at the instant of time τ, and the random magnitude x on the output of A at the instant of time $t < \tau$ will be statistically independent. Condition (e) is the condition of exponential stability on the average, introduced in [8].

In agreement with [5], conditions (a) to (d) guarantee the existence and uniqueness of the solutions of Equation (2.5), if this equation is considered as an equation given in a linear region of random magnitudes z, with a norm determined in agreement with Equation (6).

The designation $\rho(t) = \| r(t, \eta(t), \xi(t)) \|$ is introduced and it is assumed that

$$h_0 = \sup_{0 \leqslant t < \infty} \rho(t), \qquad h_1 = \sup_{0 \leqslant k < \infty} \int_{kT}^{(k+1)T} \rho(t)\, dt, \qquad h_2 = \sup_{0 \leqslant k < \infty} \Big(\int_{kT}^{(k+1)T} \rho^2(t)\, dt \Big)^{1/2}$$

where k is an integer, and T is some positive number.

Theorem 2.1. Let conditions (a) through (g) and but one of the following equations be fulfilled:

$$(A) \qquad h_0 < \frac{\varepsilon\lambda}{2B}$$

$$(B) \qquad h_1 < \frac{\varepsilon}{2B} e^{-\lambda T} (1 - e^{-\lambda T})$$

$$(C) \qquad h_2 < \frac{\varepsilon}{2B} \left(\frac{2\lambda}{e^{2\lambda T} - 1} \right)^{1/2} (1 - e^{-\lambda T})$$

If $z(t)$ is a random solution of system (2.5) determined by the condition $\| z(0) \| < \epsilon/2B$, then at $t > 0$ the inequality $\| z(t) \| < \epsilon$ holds, and, further, there exists t_0 such that $\| z(t) \| < \epsilon/2B$ for $t > t_0$.

The proof of the theorem follows. Obviously, the formula of Cauchy is justified in the given case, in agreement with which

$$z(t) = W(t, 0) z_0 + \int_0^t W(t, \tau)(R(z, \tau) + r(\tau))\, d\tau$$

where z_0 is the original random vector and for brevity of notation the random parameters $\eta(t)$ and $\xi(t)$ are omitted. Obviously there results

$$\| z(t) \| \leqslant \| W(t, 0) z_0 \| + \int_0^t \{\| W(t, \tau) R(z, \tau) \| + \| W(t, \tau) r(\tau) \|\}\, d\tau \quad (2.6)$$

Since $|W(t, 0)|$ and z_0 are independent random quantities, then

$$\| W(t, 0) z_0 \| \leqslant (M |W(t, 0)|^2)^{1/2} \| z_0 \| \leqslant B e^{-\alpha t} \| z_0 \|$$

Further

$$\| W(t, \tau) r(\tau) \|^2 \leqslant M (M_\tau [| W(t, \tau) |^2 | r(\tau) |^2])$$

From this and the limitation of condition (g) it follows that

$$\| W(t, \tau) r(\tau) \|^2 \leqslant M (M_\tau | r(\tau) |^2 M_\tau | W(t, \tau) |^2) \leqslant B^2 e^{-2\alpha(t-\tau)} \| r(\tau) \|$$

Analogously

$$\| W(t, \tau) R(z, \tau) \| \leqslant B e^{-\alpha(t-\tau)} \| R(z, \tau) \| \leqslant BL e^{-\alpha(t-\tau)} \| z(\tau) \|$$

Finally

$$\| z(t) \| \leqslant B e^{-\alpha t} \| z_0 \| + B \int_0^t e^{-\alpha(t-\tau)} (L \| z(\tau) \| + \rho(\tau))\, d\tau \quad (2.7)$$

Further reasoning is completely analogous to the reasoning brought out in the proof of the first part of Theorem 2.1 of [1].

Now it is assumed that the constructed random process $g(t)$ has isolated points of discontinuity of the first kind. This means that at the points of discontinuity t_k limits exist in the mean quadratic $\lim g(t)$ as $t \to t_k + 0$ and $\lim g(t)$ as $t \to t_k - 0$, but these limits do not coincide on the set of the Ω of the nonzero measure. Evidently the control carrying out the given process $x = g(t)$ must be of the form

$$u(t, \eta(t)) = g'(t) - f(g(t), t, \eta(t))$$

at points of existence of the derivative and

$$u(t, \eta(t)) = b_k \delta(t - t_k)$$

at points of discontinuity $t = t_k$. Here, b_k denotes the random vector, the components of which coincide for each value $w < \Omega$ with the magnitude of the discontinuity, corresponding to the component of the vector function $g(t)$.

Obviously the vector of distortion $r(t)$ in this case must have an analogous form, that is

$$r(t) = r^\circ(t) + \sum_k c_k \delta(t - t_k) \tag{2.8}$$

Let T be a positive number and let the quantity $\| r^\circ(t) \| = \rho_0(t)$ be integrable over each interval $[kT, (k + 1)T]$ $(k = 0, 1, 2, \ldots)$.

Let

$$h_1 = \sup_{0 \leqslant k < \infty} \int_{kT}^{(k+1)T-0} \| r(t) \| dt = \sup_{0 \leqslant k < \infty} \left[\int_{kT}^{(k+1)T} \| r^\circ(t) \| dt + \sum_m \| c_m \| \right]$$

Here the second sum is extended onto those m for which the points of discontinuity $C m$ lie in the interval $[kT, (k + 1)T]$.

Utilizing the reasoning brought forth in the proof of Theorem 2.1 of the present article and Theorem 3.1 of [1], it is possible to prove that upon fulfilment of the inequality

$$h_1 < \frac{\varepsilon}{2B} e^{-\lambda T} (1 - e^{-\lambda T})$$

the solution $z(t)$ of system (2.5), determined by the condition $\| z(0) \| < \epsilon/2B$, does not exceed, for $t > 0$, the limits of the area $\| z \| < \epsilon$. It is also possible to prove that such a t_0, for $t > t_0$, will have $\| z(t) \| < \epsilon/2B$.

3. Now consider the case in which the programmed process is periodic. The random function $\phi(t)$ is called periodic if there exists a period T such that

$$\| \varphi(t + T) - \varphi(t) \| = (M | \varphi(t + T) - \varphi(t) |^2)^{1/2} = 0$$

for any t. The random vector function is called periodic if its components are periodic random functions of one and the same period. The periodicity of a random matrix is determined analogously to the requirement of the periodicity of its elements. It should be noted that the indicated definition of the periodicity of a random function is equivalent to the requirement of the periodicity of almost all realizations of the function.

It is proposed now that in Equation (1.1) the function $f(x, t, \eta(t))$ is a periodic random function t of period T. Further, let the approximate random process $g(t)$ also be a periodic vector function of the same period. In this case, it is reasonable to select the control $u(t, \xi(t))$ to be periodic, and this means that the distortion $r(t, \eta(t), \xi(t))$ is also periodic of period T. Therefore, in system (2.5) let the matrix $A(t, \eta(t))$ and the functions $R(z, t, \eta(t))$, $r(t, \eta(t), \xi(t))$ be periodic of period T.

Theorem 3.1. Let there be fulfilled conditions (a) through (g) and one of the conditions

$$(A) \qquad \rho_0 = \sup_{0 \leqslant t \leqslant T} \rho(t) < \frac{\varepsilon\lambda}{2B^2}$$

$$(B) \qquad \rho_1 = \int_0^T \rho(t)\, dt < \frac{\varepsilon}{2B^2} e^{-\lambda T}(1 - e^{-\lambda T})$$

$$(C) \qquad \rho_2 = \left(\int_0^T \rho^2(t)\, dt\right)^{1/2} < \frac{\varepsilon}{2B^2}\left(\frac{2\lambda}{e^{2\lambda T} - 1}\right)^{1/2}(1 - e^{-\lambda T})$$

The following statements are then valid:

1) Any solution $z(t)$ of the system of equations (2.5), determined by the condition $\| z(0) \| < \epsilon/2B$ for $t > 0$, is limited to the region $\| z \| < \epsilon$.

2) There exists in the region $\| z \| < \epsilon$, for $t > 0$ a periodic solution, asymptotically stable in the mean quadratic [8], $z^\circ(t)$, such that from $\| z(0) - z^\circ(0) \| < \epsilon/2B$ there follows $\lim \| z(t) - z^\circ(t) \| = 0$ as $t \to \infty$.

The proof of the first part of the theorem follows from Theorem 2.1; the second part is proved exactly as the second part of Theorem 2.1 of [1], if the remarks made in the proof of Theorem 2.1 are noted.

Now suppose that the approximating process $g(t)$ again has isolated points of discontinuity of the first kind. Let

$$\rho_1 = \int_0^T \| r^\circ(t) \| \, dt + \sum_k \| c_k \|$$

where the random function $r^\circ(t)$ and the random vector c_k have the same meaning as in (2.8); the sum in the second component is extended over those values of k for which the point of discontinuity lies in the interval $[0, T]$. It is not difficult to prove that in this case the fulfilment of the inequality

$$\rho_1 < \frac{\varepsilon}{2B^2} e^{-\lambda T} (1 - e^{-\lambda T})$$

involves in itself the proof of both statements of Theorem 3.1.

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